The Tri-color Urn Process

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Abstract

An urn sampling process is introduced involving three colors. A derandomization procedure is applied to the process to generate a rotor-router model. After defining the concept of alternation between two consecutive routers, we present an inductive proof on the alternation of two consecutive rotors.

1 Introduction

The tri-color urn sampling process that we investigated had three colored balls. By convention, the colors are white, black, and grey. If a black ball is drawn, we replace the black ball in the urn and add two additional black balls to the urn. If a white ball, we follow a similar procedure, this time adding two additional white balls. If a grey ball is drawn, we replace the grey ball in the urn and add two more balls, this time a white ball and a black ball.

Let n_b and n_w denote the number of black balls and white balls respectively. The probability of drawing a particular color can be calculated as follows:

$$Pr(black) = \frac{1}{n_b}$$

$$Pr(white) = \frac{1}{n_w}$$

$$Pr(grey) = \frac{1}{n_b + n_w + 1}$$

Here we assume that the initial configuration is one black, one grey, and one white ball. Using this condition and the above probabilities, we can sketch a probability tree diagram for the process. We note in passing that this process has the Markov property since any future state of the system depends only on the previous state. The tricolor urn process is a variant on the Pólya urn process. Pólya's urn differs from the tricolor process in that the initial configuration omits the grey ball, and only one ball of the same color is added after returning the sampled ball.

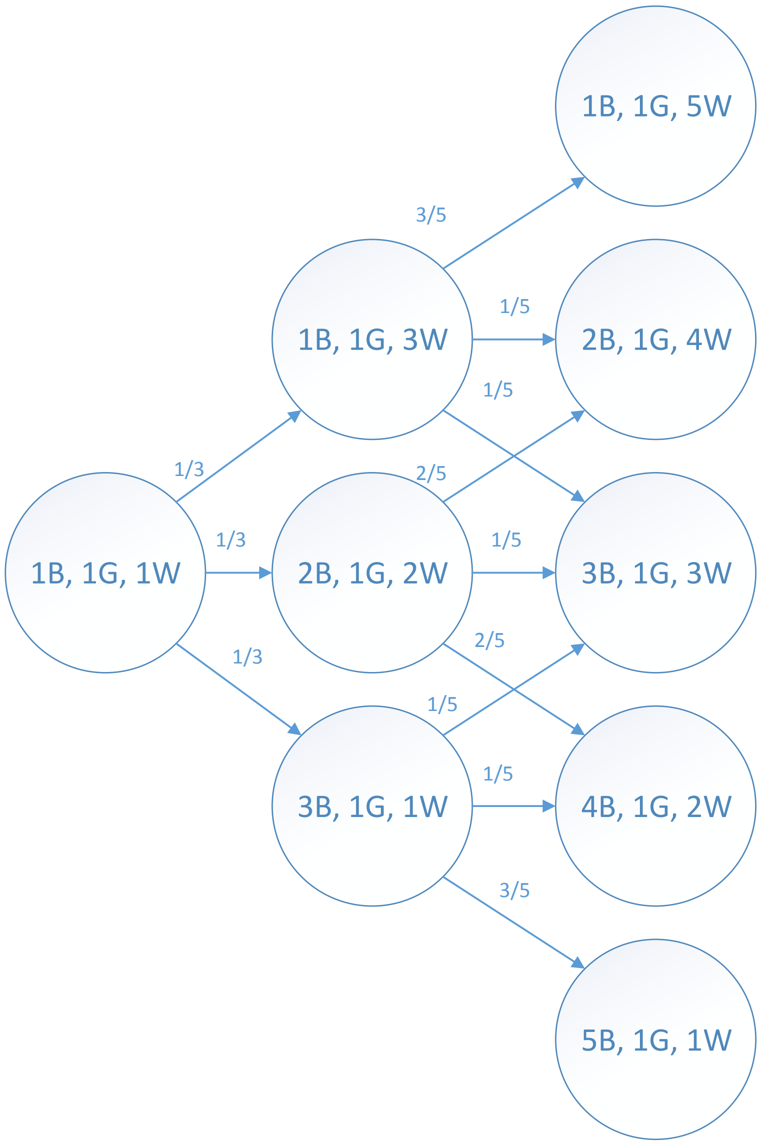

Figure 1: *Probability Tree Diagram for the Tricolor Urn Process*

2 Derandomization

The urn process has a derandomized analogue which we will call the *Tricolor Board*. This is the infinite set of points $(i, 1, j)$ where $i, j \in Z^{>0}$. We can express the first three levels as follows:

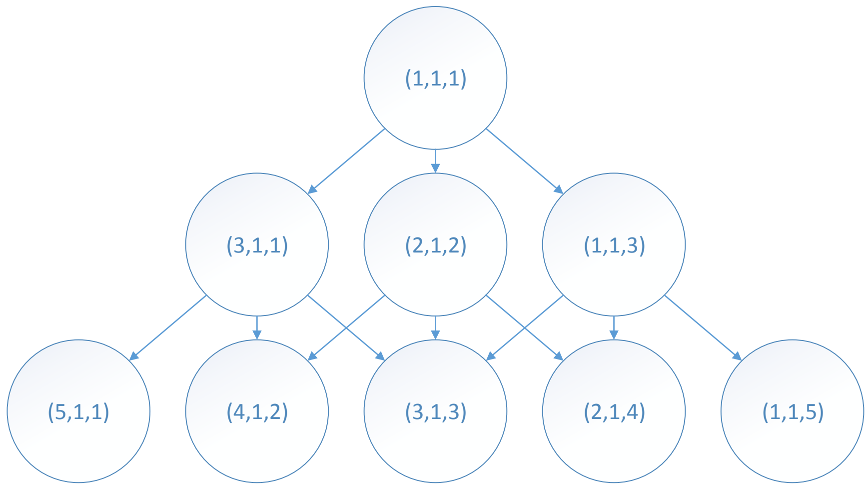

Figure 2: *First Three Levels of the Tricolor Board*

Particles are fed successively from the top of the board and descend towards the bottom via movement rules defined locally at each router.

Definition. *(Periodicity of a Router) Let n denote the periodicity of a router. Suppose a router has the configuration $(i, 1, j)$. Then it follows that $n = i+j+1$. For instance, the router $(1, 1, 3)$ has periodicity 5, as do all other routers at its level in the triangular array. (Observe that along each row, $i+j+1$ is constant.)*

Throughout the discussion, we assume the standard configuration. In this particular derandomization, the number of black balls at a given stage of the probability tree diagram become the numbered particles (modulo the periodicity of the router) that pass down and to the left in the corresponding router. Similarly, the number of white balls at a given stage become the numbered particles that pass down and to the right in the corresponding router. When the grey ball is drawn during sampling, the corresponding particle in the derandomization will move to the router directly below the current one.

Consider the mth particle. At the router $(i, 1, j)$, also denoted as L^iCR^j, the mth particle will move to the router to the left if m lies in $1, ..., i$ (modulo $i+j+1$), to the router in the center if $m = i+1$ (modulo $i+j+1$), and to the router to the right if m lies in $i+2, ..., i+j$ (modulo $i+j+1$).

3 Alternation Property

Definition. *(Arrival Time) Let $x_1, x_2, ...$ and $y_1, y_2, ...$ be the arrival times of consecutive particles at router X and router Y respectively. Two routers (X, Y) are an alternation pair if $x_1 < y_1 < x_2 < y_2 < ... < x_n < y_n < ...$*

After derandomizing the Pólya urn using a standard configuration, Einstein, Propp, and Holroyd (forthcoming) demonstrated that any two consecutive routers in the Pólya board are an alternation pair. We considered whether the same property held for the Tricolor Board, which was shown to be the case.

Theorem. *Every two consecutive routers in the Tricolor Board are an alternation pair.*

Proof. We consider first the base case where the router pairs have periodicity $n \leq 7$. This occurs in the following upper section of the triangular array:

(1,1,1)
(3,1,1) (2,1,2) (1,1,3)
(5,1,1) (4,1,2) (3,1,3) (2,1,4) (1,1,5)

Showing alternation at these levels is left as an exercise for the reader.

Suppose $n > 7$. Consider four consecutive routers A, B, C, and D. Let $L^{k+2}CR^{n-k-3}$, $L^{k+1}CR^{n-k-2}$, $L^{k}CR^{n-k-1}$, and $L^{k-1}CR^{n-k}$ be the standard configuration for A, B, C, and D respectively. Here $0 < k < n-2$. Consider E and F, the two nodes below A, B, C, and D such that all the particles passing through E and F must have first passed through A, B, C, or D. E has standard configuration $L^{k+2}CR^{n-k-1}$, and F has standard configuration $L^{k+1}CR^{n-k}$.

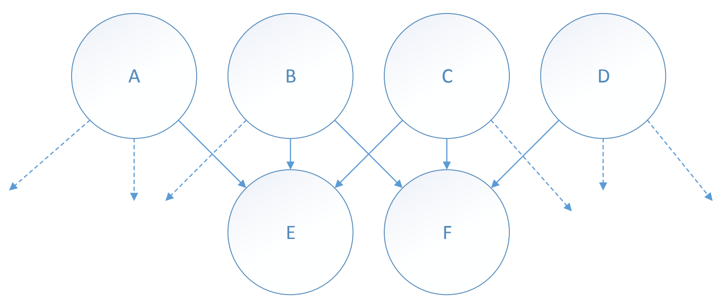

Figure 3: *4x2 System of Routers*

Note that both A and B may be fictitious, or both C and D may be fictitious. This corresponds to the left and right edge cases of the array respectively.

Assume that (A, B), (B, C), and (C,D) are alternation pairs. Then (E, F) is also an alternation pair.

Consider the particles $a_1, ..., a_n$, $b_1, ..., b_n$, $c_1, ..., c_n$, and $d_1, ..., d_n$. The particles $a_1, ..., a_{k+3}$ go off to the side, and $a_{k+4}, ..., a_n$ go to E. The particles $b_1, ..., b_{k+1}$ go off to the side, b_{k+2} goes to E, and $b_{k+3}, ..., b_n$ go to F. The particles $c_1, ..., c_k$ go to E, c_{k+1} goes to F, and $c_{k+2}, ..., c_n$ go off to the side. The particles $d_1, ..., d_{k-1}$ go to F, and $d_k, ..., d_n$ go off to the side.

The arrivals at E are equivalent to the set $\{a_{k+4}, ..., a_n, b_{k+2}, c_1, ..., c_k\}$. By the alternation of B and C, $c_1, ..., c_k < b_{k+2}$. Similarly, by the alternation of A and B, $b_{k+2} < a_{k+4}, ..., a_n$. Through the transitivity of inequality, we obtain

$$c_1, ..., c_k < b_{k+2} < a_{k+4}, ..., a_n.$$

$a_n < b_n$ since A and B are an alternation pair. Also, $b_n < c_{n+1}$ since B and C alternate. So by transitivity, $a_n < c_{n+1}$. We now having the following

$$c_1, ..., c_k < b_{k+2} < a_{k+4}, ..., a_n < c_{n+1}.$$

Because of the periodicity n of the routers, we can extend the inequality:

$$c_1, ..., c_k < b_{k+2} < a_{k+4}, ..., a_n < c_{n+1}, ...c_{n+k} < b_{n+k+2} < a_{n+k+4}, ..., a_{2n} < c_{2n+1}, ...$$

This allows us to establish the following equalities (which hold *modulo* n):

$$e_1 = c_1, e_2 = c_2, ..., e_k = c_k$$

$$e_{k+1} = b_{k+2}$$

$$e_{k+2} = a_{k+4}, ..., e_{n-2} = a_n, e_{n-1} = c_{n+1}, ...$$

At F the arrivals are equivalent to the set $\{d_1, ..., d_{k-1}, c_{k+1}, b_{k+3}, ..., b_n\}$. Since C and D alternate, $d_1, ..., d_{k-1} < c_{k+1}$. By alternation of B and C, $c_{k+1} < b_{k+3}, ..., b_n$. Invoking transitivity we obtain

$$d_1, ..., d_{k-1} < c_{k+1} < b_{k+3}, ..., b_n.$$

Using similar reasoning as above, we can show that

$$d_1, ..., d_{k-1} < c_{k+1} < b_{k+3}, ..., b_n < d_{n+1}, ..., d_{n+k-1} < c_{n+k+1} < b_{n+k+3}, ..., b_{2n}, ...$$

We obtain

$$f_1 = d_1, f_2 = d_2, ..., f_{k-1} = d_{k-1}$$

$$f_k = c_{k+1}$$

$$f_{k+1} = b_{k+3}, ..., f_{n-2} = b_n, f_{n-1} = d_{n+1}, ...$$

Having established orderings on the arrival times at the nodes E and F, we will now relate the two inequalities to build an alternation ordering between the arrival times of E and F. Since C and D alternate,

$$c_1(= e_1) < d_1(= f_1) < ... < c_{k-1}(= e_{k-1}) < d_{k-1}(= f_{k-1}) < c_k(= e_k).$$

Note that $c_k(= e_k) < c_{k+1}(= f_k)$ through the definition of arrival times. By the alternation of B and C, $c_{k+1}(= f_k) < b_{k+2}(= e_{k+1})$. By the definition of arrival times, $b_{k+2}(= e_{k+1}) < b_{k+3}(= f_{k+1})$. Finally, by the alternation of A and B, we see that $b_{k+3}(= f_{k+1}) < a_{k+4}(= e_{k+2}) < ... < a_{n-1}(= e_{n-3}) < b_{n-1}(= f_{n-3}) < a_n(= e_{n-2}) < b_n(= f_{n-2})$. By the alternation of B and C, $b_n(= f_{n-2}) < c_{n+1}(= e_{n-1})$, and the cycle repeats.

We have established the ordering

$$e_1 < f_1 < e_2 < f_2 < ... < e_n < f_n < ...$$

and therefore E and F are an alternation pair.

□

To illustrate the conjecture, let $A = (4, 1, 2)$, $B = (3, 1, 3)$, $C = (2, 1, 4)$, $D = (1, 1, 5)$, $E = (4, 1, 4)$, and $F = (3, 1, 5)$. Consider the arrival times for the nodes A, B, C, and D. For node A, $a_1, ..., a_5$ go off to the side. a_6 and a_7 go to E. For node B, $b_1, ..., b_3$ go off to the side, b_4 goes to node E, and $b_5, ..., b_7$ goes to node F. For node C, c_1 and c_2 go to E, c_3 go to F, and $c_4, ..., c_7$ goes off to the side. Finally for node D, d_1 goes to F, and $d_2, ..., d_7$ goes off to the side. All the arrivals at E are contained in the set $\{c_1, c_2, b_4, a_6, a_7\}$. Similarly, all the arrivals at F are in the set $\{d_1, c_3, b_5, b_6, b_7\}$. By the definition of arrival times, $c_1 < c_2$, and $a_6 < a_7$. Since B and and C alternate, $c_2 < b_4$. Since A and B alternate, $b_4 < a_6$. Applying the transitivity of inequality, $c_1 < c_2 < b_4 < a_6 < a_7$. Using similar reasoning, it can be shown that $d_1 < c_3 < b_5 < b_6 < b_7$.

We have established orderings on the arrival times at the nodes E and F. We will relate the two inequalities to build an alternation ordering between the arrival times of E and F.

By the alternation of C and D, $c_1 < d_1 < c_2$. Now by the alternation of C and B, $c_2 < b_2 < c_3 < b_3 < b_4$, so we can conclude that $c_1 < d_1 < c_2 < c_3 < b_4$. By the alternation of A and B, $b_4 < a_5 < b_5 < a_6 < b_6 < a_7 < b_7$. So we obtain that $c_1 < d_1 < c_2 < c_3 < b_4 < b_5 < a_6 < b_6 < a_7 < b_7$, the alternation ordering.

References

[1] A. E. Holroyd, L. Levine, K. Mészáros, Y. Peres, J. Propp, and D. B. Wilson. "Chip-Firing and Rotor-Routing on Directed Graphs." In and Out of Equilibrium 2 (2008): 331-64.

[2] D. Einstein, J. Propp, A. Holroyd. "A Derandomized Percolation Explorer" (forthcoming).